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20 Varieties of Constructivism: A Framework for Comparison

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Following the seminal influence of Jean Piaget, constructivism is emerging as perhaps the major research paradigm in mathematics education. This is particularly the case for psychological research in mathematics education. However, rather than solving all of the problems for our field, this raises a number of new ones. Elsewhere I have explored the differences between the constructivism of Piaget and that of von Glasersfeld (Ernest, 1991b) and have suggested how social constructivism can be developed, and how it differs in its assumptions from radical constructivism (Ernest, 1990, 1991a). Here I wish to begin to consider further questions, including the following: What is constructivism, and what different varieties are there? In addition to the explicit principles on which its varieties are based, what underlying metaphors and epistemologies do they assume? What are the strengths and weaknesses of the different varieties? What do they offer as tools for researching the teaching and learning of mathematics? In particular, what does radical constructivism offer that is unique? And last but not least: What are the implications for the teaching of mathematics?

CONSTRUCTION

What the various forms of constructivism all share is the metaphor of carpentry, architecture, or construction work. This is about the building up of structures from preexisting pieces, possibly specially shaped for the task. The metaphor describes understanding as the building of mental structures, and the term *restructuring*, often used as a synonym for accommodation or conceptual change, contains this metaphor. What the metaphor of construction does mean in

constructivism is that understanding is built up from received pieces of knowledge. The process is recursive (Kieren & Pirie, 1991), so the "building blocks" of understanding are themselves the product of previous acts of construction. Thus the distinction between the structure and content of understanding can only be relative in constructivism. Previously built structures become the content in subsequent constructions.

The metaphor of construction is contained in the first principle of constructivism as expressed by von Glasersfeld (1989, p. 182): "Knowledge is not passively received but actively built up by the cognizing subject." This is the principle of what I term "weak" constructivism, constituting those positions based on this principle alone.

Simple as this basic form of constructivism seems to be, it represents a very significant move from naive empiricism or classical behaviorism, for it recognizes that knowing is active, that it is individual and personal, and that it is based on previously constructed knowledge. Just getting student teachers to realize this, by reflecting on children's methods of doing mathematics or alternative conceptions in science, say, represents a significant step forward from the naive transmission view of teaching and passive reception view of learning with which many arrive. Nor is a passive reception view of learning dead among professionals or administrators in education. Many government-driven curriculum reforms, in Britain at least, assume that the central powers can simply transmit their plans and structures to teachers who will passively absorb and then implement them in "delivering the curriculum." Such conceptions and strategies are deeply embedded in the public consciousness, although it may be no accident that they also serve authoritarian powers (Ernest, 1991a; Freire, 1972).

ROMANTICISM

One danger in all forms of constructivism, which is worth mentioning here, is that it can lead to an overly child-centered, romantic progressivism. Constructivism, conceived in a loose and emotive way, can become associated with a sentimental view of the child (Walkerdine, 1984). "Discovery learning" from the 1960s onward was often bound up with a romanticism that in the end was not wholly productive for learners, and we must guard against constructivism becoming identified with this position. There is an undoubted need to let learners construct their own meanings, but also for the teacher and peers to interact with learners to negotiate a passage toward socially accepted knowledge. However, forms of discovery learning in which teachers always "funnel" learners toward predetermined solutions presuppose that the teacher is in possession of "the truth," rather than someone aware of the conventional nature of knowledge. Elsewhere I have analyzed the ideology of romanticism in progressive education in greater depth, and indicated that it often rests on an absolutist epistemology. In

such circumstances there is a perception that there are "right answers" to steer children toward (Ernest, 1991a, chapter 8). Wood, Cobb, and Yackel (1995) and Bauersfeld (1980) referred to this when embodied in classroom behaviors by the teacher as "funneling" the child toward the teacher-desired answer. The epistemological contrast between this absolutist position and the fallibilism of radical constructivism immediately reveals an area where different versions of constructivism make a significant difference in practice.

EDUCATIONAL PARADIGMS

The various forms of constructivism comprise what might be termed an educational paradigm. As such, they might be represented by:

1. An *ontology*: a theory of existence concerning the status of the world and what populates it.
2. An *epistemology* comprising (a) a theory of the nature, genesis, and warranting of subjective knowledge, including a theory of individual learning and (b) a theory of the nature, genesis, and warranting of knowledge (understood as conventional or shared human knowledge), as well as a theory of "truth."
3. A *methodology*: a theory of which methods and techniques are appropriate and valid to use to generate and justify knowledge, given the epistemology.
4. A *pedagogy*: a theory of teaching, the means to facilitate learning according to the epistemology.

Adding the fourth element (pedagogy) takes the specification of an educational paradigm beyond that which is usually admitted as an educational research paradigm, such as in Schubert (1986). However, the addition is justified by mathematics educators' shared concern with keeping educational outcomes, as well as theoretical aspects of research paradigms, in the picture.

Using this conception, I next distinguish four educational paradigms relevant to constructivism. These are information-processing theory, weak constructivism, radical constructivism, and social constructivism. Of course there are others, too. However, such paradigms differ in other ways, too. In particular, their assumed, underlying metaphor for mind and model of the world are very revealing. The mind metaphor indicates much about the epistemology of the position, as well as its pedagogy; the world model indicates much about its ontology. So I focus particularly on these aspects.

For clarity, I ought to state at the outset that my commitment is to social constructivism; what I offer also amounts to a critique of the other three positions from a social constructivist perspective, although I try also to give credit for their strengths.

Information-Processing Theory

Information processing might be regarded as one of the simpler forms of constructivism. It appears to accept von Glaserfeld's first principle, which is common to all constructivist positions, but rejects the second principle, with its far-reaching epistemological consequences (discussed later). However, I argue later that it falls short of being even a form of weak constructivism.

The information-processing paradigm is a broad church that includes the psychology of Ausubel (1968), repair theory (Brown & Van Lehn, 1982, etc.), and many of the positions adopted by researchers in cognitive science and related psychologies. It is largely based on the metaphor and sometimes the conscious model of the mind as computer. This actively processes information and data, calling up various routines and procedures, organizing memorization and retrieval of data. It can even be "heuristically" programmed, that is, modifying its outputs as it learns from experience. The computer metaphor is very fruitful because it has led to important analyses of human problem solving (Newell & Simon, 1972), and to 'buggy' error analysis (Brown & Burton, 1978), with important outcomes for the psychology of mathematics education.

The use of computers as an instructional medium is quite independent from the use of a computer metaphor of mind, and should not be confused with it. A striking example is given by Papert (1980), who is widely regarded as a constructivist. Likewise, the work of Goldin and Kaput (chapter 23, this volume), although concerned with computers and their relationship with representation, relates probably to weak constructivism and not to information-processing cognitivism.

The most common model of the world associated with information-processing theory is that of Newtonian absolute space populated by material objects (scientific realism). In general, information-processing theory represents a shift from the traditional empiricist metaphor of mind as passive to a complex mechanical (or rather electronic) metaphor of mind-as-computer, but there is usually no shift in the underlying metaphor for the world from that of absolute Newtonian space.

In summary, the ontology is the naive realism of science: The world of things we experience is out there. The things are part of an ultimate reality. The epistemology is objectivist—true knowledge of the state of affairs in the world may be possible, as is certainty in mathematical knowledge. However, where this position differs from empiricism is in its theory of learning. It represents a significant step forward from empiricism and classical behaviorism, for it recognizes that knowing involves active mental processing, it is individual, and it is based on previously acquired knowledge. Thus learning is not just a passive absorption of information; rather, it is more interactive, involving the selection, processing and assimilation of information according to the state of mind of the learner.

An important outcome of this perspective in terms of learning theory (and pedagogy) is that it accounts for student "error patterns" in mathematics (Ashlock, 1976), and similarly for misconceptions in science.

Does information-processing theory constitute a form of constructivism? It is evidently close to one, on the grounds that its account of learning seems to satisfy von Glaserfeld's first principle in that knowledge is not passively received but actively built up. However, as I indicated earlier, this principle means that the construction of knowledge is recursive, and builds on previously constructed knowledge, not passively received information or knowledge. But the metaphor of mind as computer means that at a basic level, incoming information or knowledge must simply be received by the cognizing subject, in preconstituted form, and that any complex response or elaboration that may follow its reception builds from this. This is not a recursive process of construction, but assumes instead the existence of objective, preexisting knowledge at a basic level. As one of its exponents says, alongside cognitive science, information processing theory is "the study of how humans process information, and includes the acquisition, storage and retrieval of knowledge" (Mayer, 1982, p. 3) The rhetoric of this quotation reveals the underlying presupposition that some knowledge learned by humans is information that is admitted from the outside, not constructed within. Thus, in the final analysis, information-processing theory contradicts von Glaserfeld's first principle and hence is not a form of constructivism.

Weak Constructivism

Accepting that information-processing theory is not a form of constructivism, it is clear that a weak form of constructivism can be developed from it simply by fully accepting von Glaserfeld's first principle. This is termed *weak constructivism*, and combines the principle that all individual human knowledge is constructed by each individual with the other assumptions of the information-processing paradigm already described. Thus the underlying metaphors of mind and world are almost the same. However, a difference is that the mind is an ideal "soft" computer, namely, the brain. Thus the data it processes is self-constructed, all the way down to the basic level of electrochemical nerve impulses.

There is an epistemological problem—an instability even—built into weak constructivism, for it is difficult for the dual aspects of its epistemology to coexist. On the one hand, all individual knowledge is constructed. On the other, there is a realm of objective knowledge, which, for example, would include the truths of mathematics and facts about the world. But how can such knowledge be known by any individual, if their knowledge is a personal construction? It must be the case that an individual can construct *truths*, to be able to know such knowledge. Thus the individual constructs truths about the world and of mathematics. But in this case, an individual's constructions are in fact correct representations of external states of affairs, via sense organs. This means that knowledge is constructed to match the world, or the eternal verities of mathem , and not

as a recursive construction based on previous constructions, satisfying inner constraints. In short, constructed truths can only be known as such by means of information from the world. Thus there is at the very least an antinomy—if not a direct contradiction—at the heart of weak constructivism.

Weak constructivism deals with this by being a local, as opposed to global, paradigm. That is, it accepts traditional epistemology concerning knowledge, and only tries to account for the knowledge representations of individuals. This is quite legitimate, for not all theories can be theories of everything, and we cannot simultaneously make everything the object of our inquiry.

Another possible way out of this dilemma for weak constructivism is to acknowledge that there is a pregiven world of persons, objects, and conventional knowledge (after all, denying this is problematic), but to adopt an agnostic, tentative position about our knowledge of this world. This is certainly the position of virtually all schools in the modern philosophy of science. Time and again our best theories of the world are shown to be false (Popper, 1959). This is then a defensible position. It is also appears close to that adopted by many of the other contributors to this volume, such as Booker, Goldin and Kaput, and Herscovics. It enables learner's constructions of meanings to be problematized, without having to raise larger ontological and epistemological issues. It does, however, leave the issue of the nature and status of mathematical knowledge unanswered.

Radical Constructivism

Although it originates with Piaget and is anticipated by Vico, in its modern form radical constructivism has been most fully worked out in epistemological terms by von Glasersfeld, in a series of publications over the past 15 years. In methodological terms, the leading figure in the area of mathematics education has perhaps been Steffe (1991).

Definitionally, radical constructivism is based on both the first and second of von Glasersfeld's principles. The second profoundly affects the world metaphor, as well as that of the mind: "The function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality" (von Glasersfeld, 1989, p. 182). Consequently, "From an explorer who is condemned to seek 'structural properties' of an inaccessible reality, the experiencing organism now turns into a builder of cognitive structures intended to solve such problems as the organism perceives or conceives" (von Glasersfeld, 1983, p. 50).

Although it is controversial, and the caveat is needed that there are a number of different forms of radical constructivism (Ernest, 1991b), my claim is that the underlying metaphor for the mind or cognizing subject is that of an organism underlying evolution, patterned after Darwin's theory, with its central concept of the "survival of the fit (or fitter)." This is indicated in Piaget's notion of adaptation to the environment and his explicit discussion of cognitive evolution, such as in Piaget (1972).

According to the evolutionary metaphor, the cognizing subject is a creature with sensory inputs, furnishing data that is interpreted (or rather constructed) through the lenses of its cognitive structures; it comprises also a collection of those structures all the while being adapted, and a means of acting on the outside world. The cognizing subject generates cognitive schemas to guide actions and represent its experiences. These are tested according to how well they "fit" the world of its experience. Those schemas that fit are tentatively adopted and retained as guides to action. Cognition depends on an underlying feedback loop.

Thus on the one hand, there is an analogy between the evolution and survival of the fitter of the schemas in the mind of the cognizing subject and the whole of biological evolution of species. Schemas evolve, and through adaptation come to better fit the subject's experienced world. They also split and branch out, and perhaps some lines become extinct. On the other hand, the organism itself and as a whole is adapting to the world of its experiences, largely through the adaptation of its schemas.

It is difficult to isolate the underlying model of the world of this position, for it is implicated in that of mind. It is experienceable but not knowable in any ultimate sense, just as the Kantian world of "phenomena" floats above the unattainable substratum of the "noumena." It is like the environment or world surrounding an animal: It is real and resists and constrains the animal, but is not known by the animal (including humans) over and above the ways that the animals schema's fit or fail to fit the world. No match between these schemas and the world is possible; nor could it be verified, if it did exist.

Overall, radical constructivism is neutral in its ontology, making no presuppositions about the existence of the world behind the subjective realm of experience. The epistemology is whole-heartedly fallibilist, skeptical, and antiobjectivist. The fact that there is no ultimate, true knowledge possible about the state of affairs in the world, or about such realms as mathematics, follows from the second principle, which is one of epistemological relativity. As its name implies, the theory of learning is radically constructivist, with all knowledge being constructed by the individual on the basis of its cognitive processes in dialogue with its experiential world. The pedagogy is multifaceted, but at its heart lies sensitivity to individual construction.

Radical constructivism is a rich theory that is giving rise to a whole body of fruitful and innovative research, as this volume indicates. Indeed, it is very important in mathematics and science education, where it might be said to represent the pretending if not yet ruling epistemological theory. Almost uniquely, it represents an educational paradigm—a research program, even—that has been fully developed in that the ontological, epistemological (in both senses), methodological, and pedagogical dimensions have all been extensively treated in the recent literature.

Of course such praise is all the more reason that radical constructivism should be subjected to serious critical scrutiny. One central criticism that might require a

clearer exposition of the relevant aspects of the position, or some revision, is as follows. The account of the cognizing subject emphasises its individuality, its separateness, and its primarily cognitive representations of its experiences. Its representations of the world and of other human beings are personal and idiosyncratic. Indeed, the construal of other persons is driven by whatever representations best fit the cognizing subject's needs and purposes. None of this is refutable. But such a view makes it hard to establish a social basis for interpersonal communication, for shared feelings and concerns, let alone for shared values. By being based on the underlying evolutionary metaphor for the mind there is a danger that interpersonal relations are seen as nothing but competitive, a version of the "law of the jungle." After all, this is but another way of phrasing "the survival of the fit." Yet society and its functions, in particular education, depend on articulated and shared sets of concerns and values—values that are most evidently subscribed to by radical constructivists themselves. Thus the paradigm needs to accommodate these issues by balancing knowing with feeling, and acknowledging that all humans start as part of another being, not separate.

Von Glasersfeld has shown in his treatments of aspects of radical constructivism that it is possible to elaborate the position extensively to answer if not rebut much of this criticism. Each individual's knowledge of other persons, and hence, mediately, the realm of the social, can be consistently construed as constructions of the individual knower. Such an epistemology is self-consistent, and does not fall prey to facile critiques. Similarly, being ontologically neutral, radical constructivism is not solipsistic, as some critics have claimed. Nevertheless, it does seem to put up impenetrable barriers between individuals, and between individuals and the social world. Weak constructivism, by adopting a less stringent epistemology, permits both knowledge and morality to enter from the world. No such ingress is easily countenanced by radical constructivism.

A number of researchers, such as Confrey (1995), Cobb (chapter 1, this volume), and Steffe (chapter 27, this volume), are showing how radical constructivism can be elaborated to incorporate some of the social aspects and insights that a strict formulation seems to shut out. So it might be termed a progressive research program, in the sense of Lakatos (1970).

Social Constructivism

Social constructivism regards individual subjects and the realm of the social as indissolubly interconnected. Human subjects are formed through their interactions with each other (as well as by their individual processes). Thus there is no underlying metaphor for the wholly isolated individual mind. Instead, the underlying metaphor is that of *persons in conversation,* comprising persons in meaningful linguistic and extralinguistic interaction and dialogue (Ernest, 1994; Harré, 1989). Mind is seen as part of a broader context, the "social construction of

meaning." Likewise, the social constructivist model of the world is that of a socially constructed world that creates (and is constrained by) the shared experience of the underlying physical reality. The humanly constructed reality is all the time being modified and interacting to fit ontological reality, although it can never give a "true picture" of it.

Adopting persons in conversation as the underlying metaphor of social constructivism gives pride of place to human beings and their language in its account of knowing. Following the seminal work of Wittgenstein, Vygotsky, symbolic interactionism, and activity theory, language is regarded as the shaper of, as well as being the summative product of, individual minds. Increasing attention is being given to the impact of language in much psychological research in the psychology of mathematics education, such as the cognitive role of such linguistic features as metonymy and metaphor. It is increasingly recognized that much instruction and learning takes place directly through the medium of language. Even manipulative or enactive learning, emphasised by Piaget and Bruner, takes place in a social context of meaning and is anyway mediated by language and the associated socially negotiated understandings (as Donaldson, 1978, and others have shown).

In summary, the social constructivist research paradigm adopts a modified relativist ontology (there is a world out there supporting the appearances we have shared access to, but we have no certain knowledge of it). It is based on a fallibilist epistemology that regards "conventional knowledge" as that which is "lived" and socially accepted. The associated learning theory is constructive (in the sense shared by sociologists such as Schütz, 1972, and Berger and Luckman, 1966, as well as constructivists), with an emphasis on the essential and constitutive nature of language and social interaction. The methodology is eclectic but recognizes that all knowing is problematic, and that there is no privileged vantage point. Likewise the pedagogy is eclectic, aware of the interactive and inseparable effects of the micro and macro social contexts, and the internal construction of self, beliefs, and cognitions.

COMPARISON

The metaphors for the mind and models of the world of the different paradigms are summarized in Fig. 20.1. One of the major distinctions lies in the underlying metapho. of the world (and the concomitant epistemology). The first two paradigms are based on the model of scientific realism/Newtonian absolute space, for the world and its existents. With it comes an absolutist epistemology and a neopositivist paradigm of research. This locates the "problematic" of epistemology exclusively in the immediate object of inquiry, that is, in the mind of the learner. Thus these research paradigms do not require any reflexivity or doubts about the

TYPE OF CONSTRUCTIVISM	*METAPHOR FOR THE MIND*	*MODEL OF THE WORLD*
INFORMATION-PROCESSING CONSTRUCTIVISM	Computer, unfeeling thinking machine	Newtonian Absolute space with physical objects (Scientific Realism)
WEAK CONSTRUCTIVISM	'Soft' computer (brain-as-machine)	Newtonian Absolute space with physical objects (Scientific Realism)
RADICAL CONSTRUCTIVISM	Evolving, adapting, isolated biological organism	Subject's private domain of experience
SOCIAL CONSTRUCTIVISM	Persons in conversation	Socially constructed, shared world

FIG. 20.1. Constructivist metaphors for mind and world-models.

researcher's constitutive role in knowledge and meaning making. This is a stance that is increasingly questionable in the social sciences.

In contrast, the last two paradigms do not regard the world as something that can be known with any certainty. They problematize the whole relationship between the knower and the known, and accept that no certain knowledge is attainable by humans. This humility with regard to epistemology, knowledge, and the results of the models employed in research process resonates with much of current thinking in philosophy, the humanities, and social sciences. However, it does mean that neither research paradigm nor methodology can be employed mechanically in the quest for knowledge, but that every such approach is fraught with epistemological difficulties and stands in need of justification.

Another important feature that emerges from Fig. 20.1 is that of the varying complexity of the underlying metaphors of mind. The earlier metaphors (following the order presented) offer an inanimate or simplistic model of mind. Simplifying assumptions are essential in science, but so too is the recognition of complexity. If the metaphor of mind adopted accords the cognizing subject something less than human status, there is a risk of neglecting the richness of human thought, feeling, values, reflection, planning, purposes, and goals.

Some of the metaphors of mind and world discussed earlier can be said to be leftovers from successful scientific theories of the past (as Bachelard, 1934, pointed out). By uncovering them, the outcome may be to clear the path to a better evaluation of those theories and to buttress the more powerful and practical forms of constructivism.

Finally, it is clear from the foregoing discussion that one of the crucial

	ABSOLUTIST EPISTEMOLOGY	*FALLIBILIST EPISTEMOLOGY*
INDIVIDUAL FOCUS ALONE	INFORMATION-PROCESSING CONSTRUCTIVISM WEAK CONSTRUCTIVISM	RADICAL CONSTRUCTIVISM
INDIVIDUAL AND SOCIAL FOCUS		SOCIAL CONSTRUCTIVISM

FIG. 20.2. Classification by epistemology and social/individual focus.

differences between paradigms is in a paradigm's underlying epistemology—whether it is absolutist or fallibilist. Figure 20.2 shows the paradigms classified according to whether the epistemology is fallibilist or absolutist. An increasing number of philosophers (e.g., Rorty, 1979; see also Ernest, 1991a) are adopting a fallibilist position in epistemology, and to mind it is a very great shared strength of radical and social constructivism that they support this position. In addition, another important aspect is shown, the human focus, be it individual alone or a combination of individual with social.

I have not discussed positions, such as the that of "situated learning," that seem to have a wholly social focus; see, for example, Lave and Wenger (1991). However, there are aspects of them that might be consistently combined with social constructivism, and perhaps also with weak constructivism.

Piagetian constructivism seems to emphasize internal cognitive processes at the expense of social interaction in the learner's construction of knowledge (although as always, in Piaget's rich oeuvre it is possible to find discussion of the import of both language and social interaction). However there is a need for constructivism to accommodate the complementarity between individual construction and social interaction. Von Glasersfeld (1995) has shown that mathematical knowledge is taken as shared through agreed rules and conventions, explicitly opening the door to the influence of social interaction. But further elaboration of radical constructivism is needed to recognize the fundamental implication of the social in the construction of the individual (and perhaps to rediscover the notion of "shared" in the "taken as shared").

In my view radical constructivism can be seen to be part of a "progressive research programme" (Lakatos, 1970), evolving to better describe the broad range of phenomena involved. Radical constructivism is adapting to accommodate the criticism it has met, especially concerning its possible neglect of the social and cultural dimensions of learning and knowledge. This development circumvents some of the criticism offered here. Nevertheless, important differences in the underlying model of the mind remain: wholly individual, or a

combined individual–social view. I suspect that in further research and debate in the field this difference will prove conclusive.

PEDAGOGICAL IMPLICATIONS

Ultimately, the import of an educational paradigm concerns its implications for practice, notably in pedagogy. However, in my view, there is little in any pedagogy that is either wholly necessitated or wholly ruled out by the other elements of an educational paradigm. (Goldin, 1990, also makes this point with regard to constructivism.) This is due to the fact that pedagogy is predicated on a set of values, those reflected in the following questions. What are the aims of education? What selection from the stock of cultural knowledge is valuable to teach? (Here again, I pause to consider whether radical constructivism is even able to pose this question.) What forms of human organization and interaction fit with the values? What view of the child or person, with what rights and powers, is associated with the values? A pedagogy is merely a theory of techniques for achieving the ends of "communicating" or offering the selected knowledge or experiences to learners in a way consistent with these values. The other elements of a paradigm are consistent with a wide range of pedagogical approaches, so the pedagogy of the paradigm is likely to be eclectic.

All four of the educational paradigms considered earlier suggest as pedagogical emphases the need and value for:

1. Sensitivity toward and attentiveness to the learner's previous constructions.
2. "Diagnostic" teaching attempting to remedy learner errors and misconceptions, with perturbation and cognitive conflict techniques as part of this.
3. Attention to metacognition and strategic self-regulation by learners.
4. The use of multiple representations of mathematical concepts.
5. Awareness of the importance of goals for the learner, and the dichotomy between learner and teacher goals.
6. Awareness of the importance of social contexts, such as in the difference between folk or street mathematics and school mathematics (and an attempt to exploit the former for the latter).

Beyond these, a number of further pedagogical emphases are more strongly suggested by radical and social constructivism than by the other two paradigms.

1. Knowledge as a whole is problematized, not just the learner's subjective knowledge, including mathematical knowledge and logic.
2. Methodological approaches are required to be much more circumspect and reflexi[illegible] as there is no "royal road" to truth or near truth.

3. The focus of concern is not just the learner's cognitions, but the learner's cognitions, beliefs, and conceptions of knowledge.
4. The focus of concern with the teacher and in teacher education is not just with the teacher's knowledge of subject matter and diagnostic skills, but with the teacher's beliefs, conceptions, and personal theories about subject matter, teaching, and learning.
5. Although we can tentatively come to know the knowledge of others by interpreting their language and actions through our own conceptual constructs, the others have realities that are independent of ours. Indeed, it is the realities of others along with our own realities that we humans strive to understand, but we can never take any of these realities as fixed.

Finally, a number of further pedagogical emphases are suggested by social constructivism and not by radical constructivism.

1. How is the mind of the learner formed by social interaction? Is the outcome a complex that is very much linked to the specificity of context? How does this impact on the learner's conceptions and activities in the classroom?
2. The focus of concern goes beyond learner cognitions and beliefs, to include her affect and the context-bound nature of her thought.
3. An awareness of the social construction of knowledge suggests a greater pedagogical emphasis on discussion, collaboration, negotiation and shared meanings.
4. As a social construct, mathematical knowledge is irrevocably bound up with texts and semiosis. How can this insight be accommodated in the theories and practices of mathematics education? (This last question indicates the current direction of my research, as hinted in Ernest, 1993.)

CONCLUSION

Because I have been standing outside of a range of educational research paradigms (although admitting that my sympathies lie with one of them), it seems appropriate to conclude with some reflections on my position as commentator. I have offered a schema or comparative framework, and as with any such categorization, the following question arises. How much of the differences and weaknesses indicated in the various educational paradigms is the construct of the schema or framework for comparison? It is clear that there are significant differences between the positions of information-processing theory, weak constructivism, radical constructivism, and social constructivism in terms of the categories distinguished above. However, there is a question to be asked. How neatly can any of the positions of the contributors to this volume be fitted into these categories? I know that I am sometimes uncomfortable when the complexi[illegible] my own

position is fitted into the Procrustean bed of someone else's comparative framework. One cynical perspective of an aspect of what I offer is that it is a categorization of positions by "epistemological correctness": That is, how fallibilist are the epistemologies offered? This, given my own espousal of fallibilism (Ernest, 1991a), would indeed be ironic—if not self-inconsistent. Just as we (I presume to speak for myself and the others whose views I have commented on) stress the need to understand learner conceptions in their own terms, perhaps educational paradigms also need to be understood in their own terms, too.

Some of the researchers whose work I have identified as weakly constructivist (e.g., Goldin & Kaput, chapter 23, this volume) are making important progress in accommodating the vital issues of representation and semiosis in their educational research, without the inhibitions and obstructions that a radical constructivist position can impose. It is also clear that a number of researchers in the radical constructivist tradition (e.g., Confrey, 1995; Steffe & Tzur, 1994; Wood, Cobb, & Yackel, 1995) are working to accommodate the social aspects of the teaching and learning of mathematics within their positions, and the accommodations are somewhat altering their research paradigm. One well-known social constructivist (Bauersfeld, 1995) is trying to preserve some the strengths of the radical constructivism while working to overcome its weaknesses through admitting the social dimension. How well does he fit with what I have labeled was social constructivism? There is also the question of fruitfulness. Which of the preceding educational research paradigms are providing the greatest pool of empirical knowledge and insights into children's learning of mathematics? Surely this is one of the most important questions to be asked, one that I have scarcely touched upon.

What I have done is to lay bare certain figures of my own conceptual framework. If it suggests certain areas where educational paradigms need to be sharpened up, or if it stimulates further constructive debate about the comparison of such paradigms then it has proved its worth. In all humility I can claim no more.

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21 The Construction of Conceptual Schemes in Mathematics

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> *Often, the exclusively behavioral characterization of desirable learning outcomes leads educators to rely on the teaching of discrete, disconnected skills in mathematics, rather than on developing meaningful patterns, principles, and insights.*
>
> (Goldin, 1990, p. 36)

This remark by Goldin reflects a major difference between an exclusively behaviorist approach and a constructivist view of the acquisition of knowledge. Constructivism avoids reductionist attempts to constrict all learning to the appropriation of parcels of unrelated knowledge, and provides instead a structured, organized perspective.

Inherent to a constructivist outlook is the assumption that any learning of higher order concepts involves some kind of integration by the learner into his or her existing cognition. The new knowledge needs to find some anchor points in the learner's cognition in order to maintain some cognitive continuity and relevance. Piaget's theory of equilibration (Ginsburg & Opper, 1979) describes two complementary learning processes: that of assimilation, enabling the learner to fit new knowledge into his or her existing cognitive structure, and that of accommodation, involving the reorganization and expansion of such a cognitive structure.

From an instructional viewpoint, constructivism involves some coherent organization of the knowledge to be acquired by the student. Fundamental mathematical notions, even those as elementary as the number concept, cannot be described by using classical concept formation theory, because the[y] cannot be defined in terms of attributes or in terms of examples and nonexamples. Instead